ISSN 2347-7644

Sadakath

A Research Bulletin

UGC APPROVED JOURNAL NO. (64130). INDEXED IN GOOGLE SCHOLAR

Volume VI **Number 1** **July 2018**

Published by

Sadakathullah Appa College
(Autonomous)
(Reaccredited by NAAC at an 'A' Grade with a CGPA of 3.40 in the III Cycle)
(An ISO 9001:2008 Certified Institution)

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Contents

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Optimal Solution of Interval Linear Programming Based on Generalized Confidence Interval Method

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Abstract

The interval linear programming (IvLP) is a method for decision making under uncertainty. One of the basic and difficult tasks in IvLP is to check whether a given point is weak optimal. In this point of view, the paper investigates IvLP problem, in which the constraints contain all, or mixed interval linear equations and inequalities with both non-negative by using generalized confidence interval method. Secondly, the paper considers optimal solutions of interval linear programming problems. Also, necessary and sufficient conditions for checking optimality are developed.

***Keywords**: Interval linear programming, Weak feasible solution Weak optimal Solution, Strong feasible solution, Strong optimal solution, Confidence Interval.*

Introduction

Systems of linear interval equations and inequalities frequently arise in practice, especially in situations where the data can be obtained by generalized confidence intervals. The generalized confidence intervals have established to be useful tools for making inferences in many practical IvLP.

The interval linear programming (IvLP) problems have been investigated by many authors, see e.g.,[1–14], among others. Rohn investigated strong solvability ofIvLP[1]. Steuerand Hladík discussed the linear programming (LP) problems with interval objective function coefficients. For this special case, Steuer [10] presented three algorithms to compute all weak optimal solutions; Hladík discussed many research papers related to interval linear programming. He proposed some polynomial time algorithms to check weak optimality of a given feasible solution [5]. Also he proposed on strong optimality of interval linear programming [8] and also he discussed transformations of interval linear systems of equations and in- equalities [12].Li and Luo et al discussed another special case: interval right-hand side linear program. Some necessary and sufficient conditions for checking weak optimality of given feasible solutions have been developed [11]. M. Soleimani-Damaneh [13] developed some methods to check some kinds of optimal solutions and strong optimal

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solutions, where only the cost vector *c* is considered in the case. Nevertheless, there are only few results on the issue of optimal solutions for a general IvLP, i.e., where the objective cost vector, the coefficient matrix and the right-hand vector are all interval vectors or interval matrices. The general polynomial method for checking weak optimality presented in [16].Weerahandi [15] has introduced the concept of a generalized pivotal quantity (GPQ) for a scalar parameter μ and using that parameter, one can construct an interval estimator for μ in situations where standard pivotal quantity based approaches may not be applicable. He referred to such intervals as generalized confidence intervals (GCI). Also, in this paper, discusses some new concepts of optimal solutions of IvLP.

Preliminary

Let us introduce some notation. An interval matrix is defined as:

$$A = [\underline{A},\ \overline{A}] = \{A \in R^{m\times n};\ \underline{A} \le A \le \overline{A}\} \text{ where } \underline{A},\ \overline{A} \in R^{m\times n} \text{ and } \underline{A} \le \overline{A}$$

If $\underline{A} = (\underline{a}_{ij})$ and $\overline{A} = (\overline{a}_{ij})$ then *A* is the set of matrices $A = (a_{ij})$ such that for all i and j

$$\underline{a}_{ij} \le a_{ij} \le \overline{a}_{ij}.$$

Similarly, let us define an interval vector as a one column interval matrix

$$b = [\underline{b}, \overline{b}] = \{b \in R^m;\ \underline{b} \le b \le \overline{b}\} \text{where } \underline{b},\ \overline{b} \in R^m \text{ and } \underline{b} \le \overline{b}.$$

The set of all *m*-by-*n* interval matrices will be denoted by R^{mxn} and these to fall *m*-dimensional interval vectors by R^m.

Let us denote by A_c and A_Δ the center and radius matrices given by

$$A_c = \frac{1}{2}(\underline{A} + \overline{A}), \qquad A_\Delta = \frac{1}{2}(\overline{A} - \underline{A}),$$

respectively. Then

$$A = [A_c - A_\Delta,\ A_c + A_\Delta]$$

Similarly, the center and radius vector are defined as:

$$b_c = \frac{1}{2}(\underline{b} + \overline{b}), \qquad b_\Delta = \frac{1}{2}(\overline{b} - \underline{b}),$$

respectively. Then

$$b = [b_c - b_\Delta, b_c + b_\Delta].$$

Let Y_m be the set of all {-1, 1} m-dimensional vectors,

$Y_m = \{y \in R^m \mid |y| = e\}$,

where e = $(1,1,1, \ldots .1)^T$ is the m- dimensional vectors of all 1's. For a given $y \in Y_m$.

let

$T_y = \text{diag}(y_1, \ldots, y_m)$

denote the corresponding diagonal matrix.

For a given interval matrix $A = [A_c - A_\Delta, A_c + A_\Delta] \in R^{m \times n}$ and for each vectors $y \in Y_m$ and each vector $z \in Y_n$, let us introduce the following matrix

$A_{yz} = A_c - T_y A_\Delta T_z$.

Similarly, for an interval vector $b = [b_c - b_\Delta, b_c + b_\Delta]$ and $y \in Y_m$, and for each vectors $y \in Y_m$, let us define vector,

$b_y = b_c + T_y b_\Delta$

whichmeans,

$$(b_y)_i = (b_c)_i + y_i(b_\Delta)_i = \left.\begin{matrix} \overline{b}_i & if\ y_i = 1 \\ \underline{b}_i & if\ y_i = -1 \end{matrix}\right\}$$

wherei =1,2,3,..., m.

A system of linear equations $Ax = b$ is called solvable if it has a solution, and feasible if it has a nonnegative solution. A system of linear interval equations $Ax = b$ is called weakly solvable (feasible) if there exists $A \in A$ and $b \in b$ such that system $Ax = b$ is solvable (feasible), and it is called strongly solvable (feasible) if the system $Ax = b$ is solvable (feasible) for all $A \in A$ and $b \in b$. In the same manner let us define the weak and strong solvability (feasibility) of a system of linear interval inequalities $Ax \leq b$.

Description of Generalized Confidence Interval

The values are seldom known exactly and have to be estimated. Therefore, we are interested in studying IvLP where it is some, or all coefficients and variables are in the form of interval. We use the method of estimation and obtain fiducial limits for the interval coefficients.

In practical studies, the data virtually the same object of interest are made by fixed (k) number of experimental entities. The i^{th} entity repeats its data n_i times, for large n_i. The entities may exhibit different within entity variances (heteroscedasticity). Here we will assume that the data follow normal distribution. We consider the following fixed effects model:

$Y_{ij} = \mu_i + \xi_{ij}$

with mutually independent errors, assumed to normally distributed with mean zero and (unknown) variance σ_i^2 , $i = 1, 2, \ldots, k$.

The task is make inference about the common meanμ, especially confidence intervals forμ, so we need an estimator of μ. Consider an unbiased estimator $\hat{\mu}$ of the common mean μ with variance $Var\ (\hat{\mu}) = \sum_{i=1}^{k} \lambda_i \sigma_i^2$, where $\lambda_i > 0$. If the variance components σ_i^2 are known then the pivot

$$Z = \frac{\hat{\mu} - \mu}{\sqrt{Var\ (\hat{\mu})}} \approx N(0, 1)$$

follows a normal distribution and its derived (1 - α) 100% confidence interval is:

$$\hat{\mu} - \mu(1-\alpha/2)\sqrt{Var\ (\hat{\mu})} \le \mu \le \hat{\mu} + \mu(1-\alpha/2)\sqrt{Var\ (\hat{\mu})}$$

whereμ(.) is quantile function of normal distribution. If the variance components σ_i^2 are unknown then we find the exact distribution of Z.

So, we want to compare some approximate confidence intervals for common mean derived from the simple t-statistic, the t-statistic with Satterthwaite's degrees of freedom, thet-statistic derived from Kenward- Roger method and by Welch's quantile approximation.

Interval derived from simple t-statistic:

The simple t-statistic *T* is given by

$$T = \frac{\overline{Y}_n - \mu}{\sqrt{Var(\overline{Y}_n)}}$$

$Var(\overline{Y}_n) = S^2/N$,

where

$$N = \sum_{i=1}^{k} n_i, \quad \overline{Y}_{i.} = n_i^{-1}\sum_{j=1}^{n_i} Y_{ij}, \quad \overline{Y}_n = N^{-1}\sum_{i=1}^{k} n_i \overline{Y}_{i.}$$

$$S_i^2 = (n_i - 1)^{-1}\sum_{j=1}^{n_i} (Y_{ij} - \overline{Y}_{i.})^2, \quad S^2 = (N-k)^{-1}\sum_{i=1}^{k} (n_i - 1) S_i^2 .$$

This statistic was derived under the assumption of the variance homogeneity and has a t- distribution with *N - k* degrees of freedom. So, the (1 - α) 100% confidence interval is,

$$\overline{Y}_n - t_{N-k(1-\alpha/2)}\sqrt{Var\ (\overline{Y}_n)} \le \mu \le \overline{Y}_n + t_{N-k(1-\alpha/2)}\sqrt{Var(\overline{Y}_n)}$$

wheret_{df}(.) is quantile function of Student's t distribution with df degrees of freedom.

Interval derived from t-statistic with Satterthwaite's degrees of freedom

The t- test, *Ts*, is given by

$$T_S = \frac{\overline{Y}_n - \mu}{\sqrt{Var\ (\overline{Y}_n)}}$$

with $Var\ (\overline{Y}_n) = N^{-1}\sum_{i=1}^{k} n_i S_i^2$.

In Satterthwaite approximated, the sum of χ^2 random variables to derive the null distribution of the statistic T_S is a *t* random variable with approximately *v* degrees of freedom:

$$\hat{v} = (\sum_{i=1}^{k} n_i\, S_i^2\,)^2 \,/\, (\sum_{i=1}^{k} (n_i - 1)^{-1}\, n_i^2\, S_i^2\,) .$$

The (1 - α) 100% confidence interval is

$$\overline{Y}_n - t_{\hat{v}(1-\alpha/2)} \sqrt{Var\ (\overline{Y}_n)} \le \mu \le \overline{Y}_n + t_{\hat{v}(1-\alpha/2)} \sqrt{Var\ (\overline{Y}_n)}$$

Welch's Quantile Approximation

Consider this probability equation:

$$\Pr\left[\, \overline{Y}_n - \mu \ < u(\xi) \sqrt{Var\ (\overline{Y}_n)}\,\right] = \xi \qquad (1)$$

If the variance components σ_i^2 are known then equation (1) holds true. If the variance components are unknown we have to estimate S_i^2. ξ is specified probability. Welch's approach was to approximate the distribution, i.e. to find such a quantile function *h*

$$\Pr\left[\overline{Y}_n - \mu < h\ (S_1^2, \dots, S_k^2,\ \xi)\right] = \xi \ (2)$$

that the equation (2) holds true. The (1 - α) 100% confidence interval is:

$$\overline{Y}_n - h\ (S_1^2, \dots, S_k^2,\ 1-\alpha/2) \le \mu \le \overline{Y}_n + h\ (S_1^2, \dots, S_k^2,\ 1-\alpha/2)$$

where the appropriated function *h* is,

$$h\ (S_1^2, \dots, S_k^2,\ \xi) = u_\xi \sqrt{\sum_{i=1}^{k} \lambda_i S_i^2} \left[1 + \frac{1 + u_\xi^2 \sum_{i=1}^{k} \lambda_i^2 S_i^4 \,/\, f_i}{4\left(\sum_{i=1}^{k} \lambda_i S_i^2\right)^2} - \frac{1 + u_\xi^2 \sum_{i=1}^{k} \lambda_i^2 S_i^4 \,/\, f_i^2}{2\left(\sum_{i=1}^{k} \lambda_i S_i^2\right)^2} + \frac{3 + 5u_\xi^2 + u_\xi^4 \sum_{i=1}^{k} \lambda_i^3 S_i^6 \,/\, f_i^2}{3\left(\sum_{i=1}^{k} \lambda_i S_i^2\right)^3} - \frac{15 + 32u_\xi^2 + 9u_\xi^4 u_\xi^4 (\sum_{i=1}^{k} \lambda_i^2 S_i^4 \,/\, f_i)^2}{32\left(\sum_{i=1}^{k} \lambda_i S_i^2\right)^4} \right]$$

where $f_i = n_i - 1$, $\lambda_i = \frac{n_i}{N^2}$, for $i = 1,2,\dots,k$.

Interval derived by Kenward Roger method

Kenward and Roger derived the method to estimate the variance of the generalized least square estimator (GLSE) and derived a test statistic about expected values.

$$T_{KR} = \frac{\bar{Y}_{\hat{\omega}} - \mu}{\sqrt{Var\ (\bar{Y}_{\hat{\omega}})}}$$

$$\text{with} \qquad Var\ (\bar{Y}_{\hat{\omega}}) = \left(\sum_{i=1}^{k} \hat{\omega}_i \right)^{-1} + 2\hat{\Lambda},$$

$$\text{where } \hat{\omega}_i = n_i / S_i^2, \qquad \bar{Y}_{\hat{\omega}} = \left(\sum_{i=1}^{k} \hat{\omega}_i \right)^{-1} \sum_{i=1}^{k} \hat{\omega}_i \bar{Y}_{i.}$$

and $\hat{\Lambda}$ is penalty derived from Kenward and Roger method. The statistic T_{KR} has a t distribution with approximately $\hat{m}$ degrees of freedom, where degrees of freedom $\hat{m}$ are derived by Satterthwaite's method. The (1 - α) 100% confidence interval is:

$$\bar{Y}_{\hat{\omega}} - t_{\hat{m}(1-\alpha/2)} \sqrt{Var\ (\bar{Y}_{\hat{\omega}})} \le \mu \le \bar{Y}_{\hat{\omega}} + t_{\hat{m}(1-\alpha/2)} \sqrt{Var\ (\bar{Y}_{\hat{\omega}})}$$

Interval Linear Programming (IvLP)

Consider the IvLP problem

$$\left.\begin{aligned} &\text{Min } c^T x \\ &\text{s.t.} Ax \ge (\le =) b (1) \\ &x \ge 0, \end{aligned}\right\}$$

where A $\in R^{m \times n}$, b$\in R^m$ and c$\in R^n$. A scenarioof IvLP (1) is a concrete realization of interval values, that is, any linear program with $A \in$ A, $b \in$ b andc $\in$ c.

Definition 1A vector $x \in R^n$ is called a weak feasible solution of the IvLP (1) if it satisfies $Ax \ge (\le,=) b$; $x \ge 0$,for some $A \in$ A, $b \in$ b.

Definition 2 Let x be a weak feasible solution to (1). It is called weakly optimal if it is optimal for some scenario of (1). (or$x$ be a weak optimal solution to (1)).

Lemma 4.1 (See [5])Let$\bar{x}$ be a feasible solution to a IvLP. A vector $\bar{x} \in R^n$ is an optimal solution of the IvLP

Min $c^T x$

s.t.$Ax \ge (\le,=) b$
$x \ge 0$,
if and only if the linear in equality system
$\underline{A}\, y \le 0$

$\overline{A}y \geq 0$ (2)

$c^T y \leq -1$

has no solution.

Thus,$\bar{x}$is an optimal for some c∈c if and only if the system (2) is unsolvable for some c∈c.

The following result holds for a special case of IvLP (interval objective function only).

Theorem 4.1 (See [5]) Avector $\bar{x} \in R^n$ is a weak optimal solution of the IvLP problem

Min $c^T x$

s.t.$Ax \geq (\leq =) b$

$x \geq 0$,

if and only if there is no solution to the linear system.

$$\begin{aligned}
&\underline{A}(x^1 - x^2) \leq 0\\
&\overline{A}(x^1 - x^2) \geq 0\\
&\overline{c}^T x^1 - \underline{c}^T x^2 \leq -1\\
&x^1,\ x^2 \geq 0
\end{aligned} \qquad (3)$$

Theorem 4.2(See [14]) Avector$\bar{x} \in R^n$ is a strong solution of *Ax*=*b*if and only if it satisfies

$$\begin{aligned}
&A_c x = b_c\\
&A_\Delta |x| = b_\Delta = 0
\end{aligned} \qquad (4)$$

Strong Optimal Solutions

In this section, let us propose the necessary and sufficient conditions for checking strong optimality of given vectors, for IvLP

Theorem 5.1 Let$\bar{x} \in R^n$. Denote F = $\{t_i \mid i = 1,\dots,p,\ \bar{x}_{ti} = 0\}$, where F be the linear form of the interval value of the decision variable.

Then $\bar{x}$ is an strong optimal solution to (1) if and only if $\bar{x}$ is a strong feasible solution to (1) and for each *A*∈A the linear system

$$\left.\begin{aligned}
&A(x^1 - x^2) = 0,\\
&(x^1 - x^2)_{t_i} \geq 0,\ i = 1,\ 2, \dots,\ p\\
&\overline{c}^T x^1 - \underline{c}^T x^2 \leq -1\\
&x^1,\ x^2 \geq 0
\end{aligned}\right\} \qquad (5)$$

has no solution.

Proof. "Only if"

Let $\bar{x}$ be an strong optimal solution to (1), then $\bar{x}$ is a strong feasible solution to (1) and for each $A \in A$, $b \in b$, $\bar{x}$ is a weak optimal solution to the following IvLP problem

Min $c^T x$
s.t.$Ax \le b$(6)
$x \ge 0$,

The feasible region of (6) at $\bar{x}$ reads

$Ax \le b$
$x_{t_i} \ge 0, i = 1,2,3,\ldots, p$ (7)

Let the matrix $E = (e_{ij})_{n \times n}$, where

$$e_{ij} = \left.\begin{matrix} 1 & i = j \in F \\ 0 & others \end{matrix}\right\}$$

Then the model (7) can be written as

$Ax \le b$
$-Ex \le 0$

Because $\bar{x}$ is a weak optimal solution to the IvLP (6),thus the associated linear system(3)

$$\begin{aligned} &\underline{A}(x^1 - x^2) \le 0 \\ &\overline{A}(x^1 - x^2) \ge 0 \\ &\overline{c}^T x^1 - \underline{c}^T x^2 \le -1 \\ &x^1, x^2 \ge 0 \end{aligned}$$

has no solution by lemma (4.1) where

$$A = \begin{pmatrix} A \\ -A \\ -E \end{pmatrix}$$

Hence

$$A(x^1 - x^2) = \begin{pmatrix} A \\ -A \\ -E \end{pmatrix}(x^1 - x^2) \le 0$$

That is, the linear system (5) is equivalent to the linear system (3), so for each $A \in A$ the linear system (5) has no solution.

"If"

Let $\bar{x}$ be a strong feasible solution to (1), then for each $A \in A$, $b \in b$, $\bar{x}$ is a feasible solution to (6). Because the linear system (5) has no solution for each $A \in A$, it is known that the associated linear system (3) has no solution from the analysis above. According to Theorem 4.1 it is known that $\bar{x}$ is a n optimal solution to (6). Hence $\bar{x}$ is a strong optimal solution to (6). This completes the proof of the theorem.

Conclusion

In the paper, we have suggested using confidence intervals for estimating interval values to IvLP. In practical point of view, confidence intervals based on t- statistic and Welch's method has very good reporting properties for almost all cases. The method based on Satterthwaite's degrees of freedom has good reporting properties whenever the number of observations in one experimental unit is sufficiently large or number of experimental units is increasing. The method based on Kenward and Roger does not have good properties for this model with small number of observations in one experimental unit. The study of very complicated system can be done with the help of this model and can be adapted to adjust the variation in the uncertain environments of real situations. Further, we discussed the optimal solutions of IvLP. Efficient methods to check strong optimality of given vectors for IvLP are developed. Some necessary and sufficient conditions for checking the optimality are also developed.

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CERTIFICATE

This is to certify that Dr/~~Sri~~/~~Smt~~/~~Miss~~. S. ANANTHALAKSHMI, Asst. Professor, Annai Hajira Women's College *has participated / presented a paper entitled* Optimization solution of interval Linear Programming *in One - day* ***NATIONAL CONFERENCE ON INNOVATIONS IN MATHEMATICS (NCIM-2018)*** ***organised by the UG, PG & Research Department of Mathematics at*** ***Sadakathullah Appa College (Autonomous), Rahmath Nagar, Tirunleveli*** **on 09.02.2018.**

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